

Nonlinear dynamics of dry friction

Franz-Josef Elmer

Institut für Physik, Universität Basel, CH-4056 Basel, Switzerland

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Abstract. The dynamical behaviour caused by dry friction is studied for a spring-block system pulled with constant velocity over a surface. The dynamical consequences of a general type of phenomenological friction law (stick-time-dependent static friction, velocity-dependent kinetic friction) are investigated. Three types of motion are possible: stick–slip motion, continuous sliding, and oscillations without sticking events. A rather complete discussion of local and global bifurcation scenarios of these attractors and their unstable counterparts is present.

Coulomb's laws [1] of dry friction have been well known for over 200 years. They state that the friction force is given by a material parameter (friction coefficient) times the normal force. The coefficient of static friction (i.e. the force necessary to start sliding) is always equal to or larger than the coefficient of kinetic friction (i.e. the force necessary to keep sliding at a constant velocity).

The dynamical behaviour of a mechanical system with dry friction is nonlinear because Coulomb's laws distinguish between static friction and kinetic friction. If the kinetic friction coefficient is less than the static one, *stick–slip motion* occurs where the sliding surfaces alternately switch between sticking and slipping in a more or less regular fashion [1]. This jerky motion leads to the everyday experience of squeaking doors and singing violins.

Even though Coulomb's laws are simple and well established (many calculations in engineering rely on these laws), they cannot be derived in a rigorous way because dry friction is a process which operates mostly far from equilibrium. It is therefore no surprise that deviations from Coulomb's laws have often been found in experiments. Typical deviations are as follows. (i) Static friction is not constant but increases with the sticking time [2, 3], i.e. the time since the two sliding surfaces have been in contact without any relative motion. (ii) Kinetic friction depends on the sliding velocity; for very large velocities, it increases roughly linearly with the sliding velocity like in viscous friction. Coming from large velocities, the friction first decreases, goes through a minimum, and then increases [3, 4]. In the case of boundary lubrication (i.e. a few monolayers of some lubricant are between the sliding surfaces) it decreases again for very low velocities (see figure 1) [5, 6]. The coefficient of kinetic friction as a function of the sliding velocity therefore has at least one extremum. The kinetic friction can exceed the static friction, but in the limit of zero sliding velocity it is still less than or equal to the static friction.

The aim of this paper is to give a rather complete discussion of the nonlinear dynamics of a single degree of freedom for an *arbitrary phenomenological* dry friction law in the sense mentioned above. This goes beyond the discussion of specific laws found in the literature [1, 3, 4, 6–9]. A phenomenological law for the friction force depends only on the

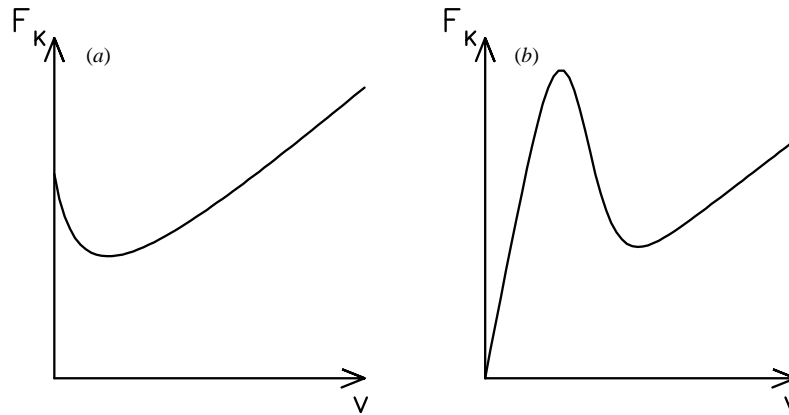


Figure 1. Schematic sketches of typical velocity-dependent kinetic friction laws for (a) systems without and (b) systems with boundary lubrication.

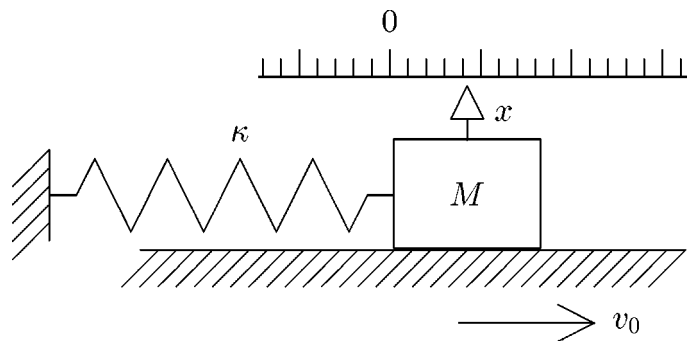


Figure 2. A harmonic oscillator with dry friction.

macroscopic degrees of freedom. This implies that all microscopic degrees of freedom are much faster than the macroscopic ones. At the end of this paper a simple argument will be given on why this assumption will not always be valid. To reveal this invalidity on the macroscopic level, it is therefore important to have a *complete knowledge of the dynamical behaviour under the assumption that this timescale separation works*.

There are two other important reasons for knowing the consequences of the different dry friction laws. (i) Friction coefficients can only be measured within an apparatus (for example the surface force apparatus [10] or the friction force microscope [11]). Below we will see that the dynamical behaviour of the whole system is strongly determined by the friction force and the properties of the apparatus. For example, stick–slip motion makes it difficult to directly obtain the coefficient of kinetic friction as a function of the sliding velocity. Thus, the influence of the measuring apparatus cannot be eliminated. (ii) Dry friction also plays an important role in granular materials [12]. An open question there is whether or not the cooperative behaviour of many interacting grains is significantly influenced by the dynamical behaviour due to modifications of Coulomb's laws.

The mechanical environment (e.g. the apparatus) of two sliding surfaces may have many macroscopic degrees of freedom. The most important one is the lateral one. Here only systems are discussed which can be well described by this single degree of freedom. Figure 2

schematically shows the apparatus. It is described by a harmonic oscillator where a block (mass M) is connected via a spring (stiffness κ) to a fixed support (see figure 2). The block is in contact with a surface which slides with constant velocity v_0 . The interaction between the block and the sliding surface is described by a sticking-time-dependent static friction force $F_S(t_{\text{stick}})$ and a velocity-dependent kinetic friction force $F_K(v)$. For the equation of motion we have to distinguish whether the block sticks or slips. If it sticks, its position x grows linearly in time until the force in the spring (i.e. κx) exceeds the static friction F_S . Thus

$$\dot{x} = v_0 \quad \text{if } |x| \leq F_S(t - t_r)/\kappa \tag{1a}$$

where $t_r < t$ is the time at which the block has stuck again after a previous sliding state. If the block slips, the equation of motion reads

$$M\ddot{x} + \kappa x = \text{sign}(v_0 - \dot{x})F_K(|v_0 - \dot{x}|) \tag{1b}$$

if $\dot{x} \neq v_0$ or $|x| > F_S(0)/\kappa$ where $\text{sign}(x)$ denotes the sign of x .

We start our investigation with Coulomb’s laws of constant static and kinetic friction. As long as $\dot{x} < v_0$ the system behaves like an undamped harmonic oscillator with the equilibrium position shifted by the amount of F_K/κ . Thus, there are infinitely many oscillatory solutions. Below we will see that some of them may survive in the case of velocity-dependent kinetic friction. The equilibrium position of the block is $x = F_K(v_0)/\kappa$. It is called the *continuously sliding state*.

Every initial state which would lead to an oscillation with a velocity amplitude exceeding v_0 leads in a *finite* time to stick–slip motion. Independent of the initial condition, the slips always start with $x = F_S/\kappa$ and $\dot{x} = v_0$. Thus, the stick–slip motion defines an attractive limit cycle in phase space. This is not in contradiction with the fact that the system behaves otherwise like an undamped harmonic oscillator. The reason for that is that a finite bounded volume in phase space is contracted onto a line if it hits that part in phase space which is defined by (1a). Stick–slip motion requires a kinetic friction F_K which is strictly less than the static one. Usually the sticking time $t_{\text{stick}} = 2(F_S - F_K)/(\kappa v_0)$ is much larger than the slipping time $t_{\text{slip}} = 2(\pi - \arctan[(F_S - F_K)v_0^{-1}(\kappa M)^{-1/2}])\sqrt{M/\kappa}$. The maximum amplitude of the stick–slip oscillation (i.e. $\max_t x(t)$) is a monotonically increasing function of v_0 which starts at F_S/κ for $v_0 = 0$. This is also true for a velocity-dependent kinetic friction force.

The unmodified Coulomb’s law leads to a coexistence of the continuously sliding state and stick–slip motion for any value of the sliding velocity v_0 . In the more general case of a velocity-dependent kinetic friction this bistability still occurs but in a restricted range of v_0 . Especially, there will always be a critical velocity v_c above which stick–slip motion disappears. This is an everyday experience: squeaking of doors can be avoided by moving them faster.

In order to be more quantitative we solve the equation of motion for a linear dependence of F_K on v , i.e. $F_K(v) = F_{K0} + \gamma v$, with $\gamma > 0$. Equation (1b) becomes the equation of a damped harmonic oscillator which can be easily solved. Instead of a continuous family of oscillatory solutions we have an attractive continuously sliding state. Stick–slip motion disappears if the trajectory with $x(0) = F_S/\kappa$ and $\dot{x}(0) = v_0$ never sticks for $t > 0$. The critical velocity $v_0 = v_c$ is defined by $x(t_{\text{slip}}) = F_{K0}/\kappa$ and $\dot{x}(t_{\text{slip}}) = v_0$. It leads to two nonlinear algebraic equations for t_{slip} and v_c . For $\gamma \ll \sqrt{M\kappa}$ the solution can be given approximately:

$$v_c = \frac{F_S - F_{K0}}{\sqrt{2\pi\gamma\sqrt{\kappa M}}} + \mathcal{O}(\sqrt{\gamma}). \tag{2}$$

The critical velocity v_c plays an important role in the discussion of the nature of stick–slip motion, because its measurement tells us indirectly something about the mechanisms of dry friction (see the discussion in [6]).

Next we discuss a general non-monotonic $F_K(v)$ like the examples shown in figure 1. The static friction F_S is still assumed to be constant. The continuously sliding state exists for all values of v_0 but it is only stable if $F'_K \equiv dF_K(v_0)/dv_0 > 0$. At an extremum of $F_K(v)$ the stability changes and a Hopf bifurcation occurs. Near the extremum and for small deviations from the continuously sliding state the dynamics of

$$x(t) - \frac{F_K(v_0)}{\kappa} = A(t)e^{i\sqrt{\kappa/M}t} + \text{c.c.} \quad (3)$$

is governed by the amplitude equation (normal form) [13]

$$\frac{dA}{dt} = -\frac{F'_K}{2M}A - \left(\frac{\kappa F''_K}{4M^2} + i \left(\frac{F''_K}{M} \right)^2 \sqrt{\frac{\kappa}{M}} \right) |A|^2 A. \quad (4)$$

If the third derivative of the kinetic friction at an extremum is positive, the Hopf bifurcation is supercritical, and in addition to the well known attractors mentioned above, another type of attractor appears. Here it is called the *oscillatory sliding state*. It is a limit cycle where the maximum velocity always remains less than v_0 . Thus the block never sticks. Its frequency is roughly given by the harmonic oscillator of the left-hand side of (1b). The second derivative of the kinetic friction is responsible for nonlinear frequency detuning. Note that the frequency of the stick–slip oscillator is usually much smaller than the frequency of the oscillatory sliding state. This oscillatory state is similar to the limit cycle of Rayleigh's equation $\ddot{u} + \epsilon(\dot{u}^3 - \dot{u}) + u = 0$ [13], in fact, Rayleigh's equation is a special case of (1b). Depending on the kinetic friction, several stable and unstable limit cycles may exist. By varying v_0 they are created or destroyed in pairs due to saddle-node bifurcations.

It should be noted that the Hopf bifurcation described by (4) is not related to the Hopf bifurcation observed by Heslot *et al* [3] which occurs in a regime (called the creeping regime) where (1a) is not applicable (see also the discussion below about the validity of dry friction laws).

An oscillatory sliding state exists only if its maximum velocity is smaller than the sliding velocity v_0 because of the sticking condition (1a). How does the interplay of the oscillatory sliding states and the sticking condition lead to stick–slip motion? In order to answer this question we calculate the backward trajectory of the point $\lim_{\epsilon \rightarrow 0} (F_K(0)/\kappa, v_0 - \epsilon)$ in accordance with (1b). Three qualitatively different backward trajectories are possible.

(1) The backward trajectory hits the sticking condition. Together they define a bounded set of initial conditions leading to non-sticking trajectories. The boundary of this set is called the special stick–slip boundary; it is not a possible trajectory but it separates between the basins of attraction of the stick–slip oscillator and the non-stick–slip attractors.

(2) The backward trajectory spirals inwards towards an unstable oscillatory or continuously sliding state. Again all initial states outside these repelling states are attracted by a stick–slip limit cycle.

(3) The backward trajectory spirals outward towards infinity, and stick–slip motion is impossible.

Two types of local bifurcations are possible: if the backward trajectory changes from case 1 to case 3 the stick–slip limit cycle annihilates with the special stick–slip boundary. For changes from case 1 to case 2 the special stick–slip boundary is either replaced by an unstable continuous or oscillatory sliding state or it annihilates with a stable continuous or oscillatory sliding state. A change from case 2 to case 3 is not possible. Figure 3 shows,

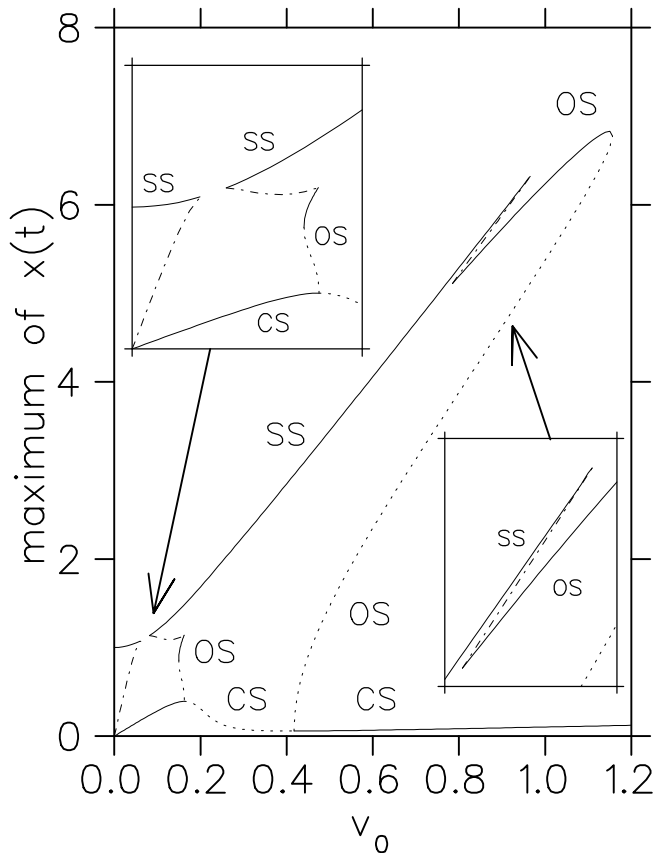


Figure 3. Typical bifurcation scenarios for a particular kinetic friction force $F_K(v)$ of the type shown in figure 1(b). The following function has been chosen $F_K(v) = v[\gamma_1 + \gamma_2 + (\gamma_2 - \gamma_1)(v - \bar{v})/\sqrt{(\Delta v)^2 + (v - \bar{v})^2}]/2$, with $\gamma_1 = 3$, $\gamma_2 = 0.1$, $\bar{v} = 0.2$, and $\Delta v = 0.05$. The results are obtained by numerically integrating the equation of motion (1). The other parameters are $F_S = 1$, $M = 40$, and $\kappa = 1$. Full (dotted) curves indicate stable (unstable) continuously sliding states (CS), oscillatory sliding states (OS), or stick-slip motions (SS). The chain curve indicates the special stick-slip boundary.

for a particular choice of $F_K(v)$, both types of bifurcations. Here the first bifurcation type occurs at $v_0 \approx 0.059, 0.082$, and 0.966 . The second type occurs at $v_0 \approx 0.162$ and 0.785 . This example shows that for increasing v_0 stick-slip motion can disappear and reappear again.

Besides the well known bistability between stick-slip motion and continuous sliding [3], multistability between one continuously sliding state, several oscillatory sliding states, and one stick-slip oscillator are possible (see figure 3). Eventually for large sliding velocities all attractors except that of the continuously sliding state will disappear because the kinetic friction has to be an increasing function for sufficiently large sliding velocities.

The strongly overdamped limit (i.e. $|dF_K(v)/dv| \gg \sqrt{\kappa M}$ for any v except in tiny intervals around the extrema) leads to a separation of timescales. From an arbitrary point (x, \dot{x}) in phase space with $\dot{x} < v_0$ the system moves very quickly into the point (x, v) where v is a solution of $\kappa x = F_K(v_0 - v)$ with $F'_K(v_0 - v) > 0$. Points on the curve $\kappa x = F_K(v_0 - \dot{x})$ with $F'_K < 0$ are unstable. They separate basins of attraction of different solutions v . After

the fast motion has decayed the system moves slowly on the curve $\kappa x = F_K(v_0 - \dot{x})$. The direction is determined by the sign of \dot{x} . It either reaches a stable continuously sliding state, or, near an extremum of F_K , it jumps suddenly to another branch of the curve or to the sticking condition. For kinetic friction laws of the form shown in figure 1(b) with v_0 between the two extrema, we get an oscillatory sliding state. It is a relaxation oscillation which may be difficult to distinguish from a stick–slip oscillation. In the case of a friction law with a single minimum at $v = v_m$ as shown in figure 1(a) we get stick–slip motion for $v_0 < v_m$ [7]. In the strongly overdamped limit any multistability disappears except near the extrema of $F_K(v)$. The experiments of Yoshizawa and Israelachvili [14] are consistent with the assumption that the system is in a strongly overdamped limit with a friction law as shown in figure 1(a) [7].

In order to discuss the influence of a stick-time-dependent static friction on the stick–slip behaviour we define a stick–slip map $x_{n+1} = T(x_n)$, where x_n is the position of the block just before slipping. For constant static friction the map reads $T(x) = F_S/\kappa$. The position just at the time of the slip-to-stick transition is defined by x_n^s . It is a function of x_n , i.e. $x_n^s = g(x_n)$, where g is usually a monotonically decreasing function. The sticking time t_n^{stick} is the smallest positive solution of

$$\frac{F_S(t_n^{\text{stick}})}{\kappa} = x_n^s + v_0 t_n^{\text{stick}}. \quad (5)$$

This defines a function $t_n^{\text{stick}} = h(x_n^s)$ which is a monotonically decreasing function due to $F'_S > 0$. Thus the stick–slip map is given by $T(x) = F_S(h(g(x)))/\kappa$. If the map has one fixed point, then stick–slip motion exists.

For $F_K = \text{constant} = F_S(0)$, stick–slip motion disappears if $v_0 > v_c = \sup_{t>0} 2[F_S(t) - F_S(0)]/(\kappa t)$. For a non-convex $F_S(t)$, the supremum occurs at a non-zero value of the sticking time leading to a saddle-node bifurcation of a stable and an unstable fixed point of the stick–slip map. At $v_0 = v_c$ the stick–slip motion has a finite amplitude, in contrast to the case of a convex $F_S(t)$ [8]. Because T is a monotonically increasing function, limit cycles or even chaos are not possible. If the slip-to-stick transition does not happen at the first time when \dot{x} becomes equal to v_0 (because of $|x_n^s| > F_S(0)/\kappa$) chaotic motion may occur [9]. In this case we get a non-monotonic T due to a non-monotonic g . Such over-shooting is only possible if $F_S(\infty)/F_S(0)$ becomes relatively large. For example, for a constant kinetic friction over-shooting occurs if $F_S(\infty)/F_S(0) > 1 + F_K(0)/F_S(0)$. For most realistic systems this condition is not satisfied. Note that the possibility of chaos is not in contradiction to the fact that the equation of motion (1) with constant F_S cannot show chaotic motion. But the retardation of F_S turns (1) into a kind of differential-delay equation.

Using phenomenological dry friction means that we treat dry friction as an element in a mechanical circuit with some nonlinear velocity-force characteristic such as, say, a diode in an electrical circuit. This treatment is justified as long as the macroscopic timescales are much larger than any timescale of the internal degrees of freedom of the interacting solid surfaces. But there is one internal timescale which diverges if the relative velocity between the surfaces goes to zero: it is given by the ratio of a characteristic lateral length scale of the surface and the relative sliding velocity. Thus, any kinetic friction law $F_K(v)$ becomes invalid if

$$v \lesssim \frac{\text{microscopic length scale}}{\text{macroscopic timescale}}. \quad (6)$$

The characteristic length scale ranges from several micrometres to several metres. It may be the size of the asperities, the size of the contact of the asperities, the correlation length

of surface roughness, or an elastic correlation length. This limitation of any dry friction law does not concern oscillatory sliding states and continuously sliding states, as long as their relative sliding velocity always stays much larger than the critical velocity (6). But the transition between sticking and sliding in a stick–slip motion may be strongly affected by the fact that just after stick-to-slip transitions and just before slip-to-stick transitions, details of the interface dynamics become important. One may expect that the importance of these details increases when the maximum slipping velocity decreases. For example, Heslot *et al* [3] experimentally found a completely different behaviour when the maximum relative sliding velocity during a slip was below the critical value (6).

In this paper the nonlinear dynamics of a harmonic oscillator sliding over a solid surface has been discussed under the assumption that dry friction can be described by a velocity-dependent kinetic friction and a sticking-time-dependent static friction. Besides the well known continuously sliding state and the stick–slip oscillator, an oscillatory sliding state without sticking has been found. All typical bifurcation scenarios of these states are shown in figure 3.

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